

Application of the Nitsche method to three industrial 3D elastoplastic frictional contact problems

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Résumé — In this note, we present how the use of an extension of the Nitsche method lends itself well to the numerical treatment of industrial cases of frictional contact between elastoplastic solids under SYSTUS/ SYSWELD 2024. The three cases presented are the self-contact of the coils of a spring, the closing/opening of a "clip", and finally the dudging of a tube.

Keywords - contact, Coulomb friction, elasto-plastic model, numerical methods

1 Introduction

The numerical simulation of multi-body contact problems involving friction and elasto-plastic material behavior is of paramount importance in mechanical engineering. These problems are characterized by severe non-linearities arising from three primary sources : the geometric constraints of impenetrability, the complex physics of frictional slip, and the non-linear constitutive response of the material itself. This combination results in a class of challenging numerical problems that are most naturally formulated as variational inequalities. The mathematical foundation for treating such deformable continuum mechanics problems in the presence of rigid obstacles was significantly advanced through the finite element method. A semi-discretization in space leads to a system of matrix equations and inequalities known as the elasto-dynamic complementarity problems or Klarbring systems. Pioneering analyses of these variational inequalities by J.J. Moreau and later by P. Ballard revealed the inherent mathematical complexity of the solutions. They demonstrated that even for elastic materials, the solutions exhibit low regularity : velocities are functions of Bounded Variation (BV), and accelerations are Radon measures. This inherent irregularity poses significant challenges for classical numerical schemes. Parallel to these theoretical developments, robust numerical algorithms were designed to solve these variational inequalities, under the assumption of small perturbations. A primary goal of these algorithms has been to regularize the contact law, thereby obtaining approximate solutions that are computationally tractable while respecting the underlying physics. Traditional approaches include the Lagrange multiplier method, which introduces additional variables to enforce the contact constraint exactly but requires the satisfaction of a discrete inf-sup condition, and the penalty method, which is simpler but inherently inconsistent and sensitive to the choice of a penalty parameter [6]. In recent decades, Nitsche's method has emerged as a powerful and elegant alternative. Originally introduced by Nitsche [10] for the weak imposition of boundary conditions, it has since been extensively applied to interface-coupled problems. Its first application to contact mechanics was presented by Wriggers et al. [17], but it was a non-consistent formulation. Another complete mathematical analysis of linearized elasticity provided later by Chouly, Hild, Renard et al. unlocked the problem with a new consistent formulation [1, 3, 4, 5, 7, 8]. The principal advantage of Nitsche's method is that it is variationally consistent, ensuring optimal convergence rates, without introducing additional degrees of freedom like the Lagrange multiplier method [9]. This eliminates the need to satisfy a discrete inf-sup condition. This advantage comes at the cost of having to evaluate the boundary traction from the continuum stresses. To enhance stability, advanced symmetric and skew-symmetric variants have been developed, the latter being stable for any positive penalty parameter. The method has been successfully extended to more complex scenarios, including finite deformation elasto-plastic contact. In this context, the approximation strategy proposed here, which implements a weak integral contact condition conceptually similar to Lagrange multipliers but within the Nitsche framework, was first developed and

validated in the open-source finite element library GetFEM [11, 12]. It has been applied to a wide range of contact scenarios, including small and large deformations for elastic and hyperelastic materials, with and without friction.

The goal of this note is to describe the application of Nitsche's method to enforce contact conditions, with or without Coulomb friction, between two elasto-plastic bodies. After that, we present how this method has been operationalized within the industrial finite element software SYSTUS/SYSWELD 2024 to solve practical engineering problems, demonstrating its effectiveness within the small deformations framework.

2 The unilateral contact problem : master-slave formulation

In this section, we present the strong (formal) and weak formulations of the studied frictional two-body contact problem with an elasto-plastic material law and linear isotropic hardening. We consider two deformable bodies Ω_α , $\alpha = 1$ or 2 ; these are domains with piecewise C^1 boundaries included in \mathbb{R}^{ndim} , where $ndim = 2$ or 3 , representing the reference configurations of two elastic bodies. The boundary Γ_1 of Ω_1 (and Γ_2 of Ω_2 respectively) is divided into three non-overlapping parts : $\Gamma_{1,C}$, the slave potential zone of contact with $meas(\Gamma_{1,C}) > 0$ (and $\Gamma_{2,C}$, respectively, with $meas(\Gamma_{2,C}) > 0$); $\Gamma_{1,N}$, the Neumann part (and $\Gamma_{2,N}$ respectively), and $\Gamma_{1,D}$, the Dirichlet part with $meas(\Gamma_{1,D}) > 0$ (and $\Gamma_{2,D}$, respectively, with $meas(\Gamma_{2,D}) > 0$).

The two bodies are subjected to volume forces $\vec{\mathbf{f}} = (\vec{\mathbf{f}}_1, \vec{\mathbf{f}}_2)$ on $\Omega_1 \times \Omega_2$, to surface loads $\vec{\ell} = (\vec{\ell}_1, \vec{\ell}_2)$ on $\Gamma_{1,N} \times \Gamma_{2,N}$ and satisfy non-homogeneous boundary Dirichlet conditions on $\Gamma_{1,D} \times \Gamma_{2,D}$, with the displacement being prescribed to the given value $\vec{\mathbf{u}}_D = (\vec{\mathbf{u}}_{1,D}, \vec{\mathbf{u}}_{2,D})$. Superscript 1 denotes the slave surface and superscript 2 denotes the master surface. $\vec{\mathbf{X}}$ is the position of a point on the slave surface $\Gamma_{1,C}$ and $\vec{\mathbf{Y}}$ is its closest point projection on the master surface $\Gamma_{2,C}$. We are interested in the displacements $\vec{\mathbf{u}} = (\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2)$ and assume small (elasto-plastic) deformations for the two bodies.

In the deformed configuration Ω_α^t , at time t , different portions of the boundary $\partial\Omega_\alpha^t$ of Ω_α^t may come into contact and interact with Ω_β^t . A non-penetration condition on the deformed contact surfaces $\Gamma_{1,C}^t$ and $\Gamma_{2,C}^t$ can be expressed with the help of a mapping function linking a point $\vec{\mathbf{X}}$ to its mapping $\Pi(\vec{\mathbf{X}})$. We denote by $\Gamma_{\alpha,C}^t \subset \Gamma_\alpha^t$ (resp. $\Gamma_{\alpha,C}^0 \subset \Gamma_\alpha^0$) the set of points $\vec{\mathbf{X}}$ (resp. $\vec{\mathbf{X}}$) in the deformed (resp. reference) configuration. Recall that points $\vec{\mathbf{X}}$ and $\vec{\mathbf{x}}$ have dimensions of $ndim$; $\vec{\mathbf{X}}$ represents Material or Lagrangian coordinates, while $\vec{\mathbf{x}}$ represents Spatial or Eulerian coordinates.

Definition 2.1 *The term small perturbation hypothesis (or even the small displacement hypothesis) comprises the assumptions of small displacements, infinitesimal transformations, and infinitesimal deformations, which enables us to proceed with the physical linearization of the constitutive law for the material. Then, the linearized strain tensor field is given by*

$$\varepsilon(\vec{\mathbf{u}}) = \frac{1}{2}(\nabla\vec{\mathbf{u}} + \nabla\vec{\mathbf{u}}^\top) \quad (1)$$

The displacements $\vec{\mathbf{u}} = (\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2)$ fulfill the following conditions for $\alpha = 1, 2$:

$$\varepsilon(\vec{\mathbf{u}}_\alpha) = \mathbf{H}_\alpha\sigma(\vec{\mathbf{u}}_\alpha) + \varepsilon_{\alpha,plas} \text{ in } \Omega_\alpha \quad (2)$$

Relation (2) describes the material law, relating the linearized strain $\varepsilon(\vec{\mathbf{u}}_\alpha)$ to the stress $\sigma(\vec{\mathbf{u}}_\alpha)$. The strain is divided into an elastic part $\mathbf{H}_\alpha\sigma(\vec{\mathbf{u}}_\alpha)$, where \mathbf{H}_α is the fourth-order symmetric elasticity tensor (compliance tensor corresponding to isotropic material that satisfies the usual uniform ellipticity and boundedness properties), and a plastic part $\varepsilon_{\alpha,plas}$.

Then, the stress tensor field $\sigma = (\sigma_{ij})_{1 \leq i, j \leq d=2 \text{ or } 3}$ is given by

$$\sigma(\vec{\mathbf{u}}) = \mathbf{H}_\alpha^{-1}(\varepsilon(\vec{\mathbf{u}}) - \varepsilon_{\alpha,plas}).$$

Consequently, the displacement $\vec{\mathbf{u}} = (\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2)$ on $\Omega_1 \times \Omega_2$ must satisfy the following set of equations,

apart from the contact condition, which will be described later :

$$\left\{ \begin{array}{ll} \text{Find } \vec{\mathbf{u}} = (\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2) \text{ satisfying} & \\ -\text{div} \boldsymbol{\sigma}(\vec{\mathbf{u}}_\alpha) = \vec{\mathbf{f}}_\alpha & \text{in } \Omega_\alpha, \\ \boldsymbol{\sigma}(\vec{\mathbf{u}}_\alpha) = \mathbf{H}_\alpha^{-1}(\boldsymbol{\varepsilon}(u_\alpha) - \boldsymbol{\varepsilon}_{\alpha, \text{plas}}) & \text{in } \Omega_\alpha, \\ \boldsymbol{\varepsilon}_{\alpha, \text{plas}}(\boldsymbol{\tau} - \boldsymbol{\sigma}(\vec{\mathbf{u}}_\alpha)) \geq 0, \forall \boldsymbol{\tau} \text{ with } \mathcal{F}_{\alpha, \text{iso}}(\boldsymbol{\tau}, |\boldsymbol{\varepsilon}_{\alpha, \text{plas}}|_F) \leq 0 & \text{in } \Omega_\alpha, \\ \vec{\mathbf{u}}_\alpha = \vec{\mathbf{u}}_{\alpha, D} & \text{on } \Gamma_{\alpha, D}, \\ \boldsymbol{\sigma}(\vec{\mathbf{u}}_\alpha) \vec{\mathbf{n}}_\alpha = \vec{\mathbf{l}}_\alpha & \text{on } \Gamma_{\alpha, N}. \end{array} \right. \quad (3)$$

The deviatoric part of a tensor $\boldsymbol{\tau}$ is denoted by $\boldsymbol{\tau}^{\text{dev}} := \boldsymbol{\tau} - \frac{1}{d} \text{tr}(\boldsymbol{\tau}) \mathbb{I}_{d \times d}$ and $|\cdot|_F$ denotes the Frobenius norm. The yield function $\mathcal{F}_{\alpha, \text{iso}}$ is defined as $\mathcal{F}_{\alpha, \text{iso}} = |\boldsymbol{\tau}^{\text{dev}}|_F - (\boldsymbol{\sigma}_\alpha^0 + \xi_{\alpha, \text{iso}} \boldsymbol{\eta})$, where $\boldsymbol{\sigma}_\alpha^0$ is the yield stress and $\xi_{\alpha, \text{iso}}$ is the isotropic hardening parameter. It is important to note that the complementarity condition ensures that plastic strain may only occur if the yield function equals zero.

Now, concerning the contact conditions, let us introduce some important definitions.

Definition 2.2 Π denotes the orthogonal projection from the slave boundary $\Gamma_{1,C}$ onto the master boundary $\Gamma_{2,C}$:

$$\Pi : \begin{array}{l} \Gamma_{1,C} \rightarrow \Gamma_{2,C} \\ \vec{\mathbf{X}} \mapsto \Pi(\vec{\mathbf{X}}) = \vec{\mathbf{Y}}. \end{array} \quad (4)$$

Remark 2.1 The operator Π is assumed to be a C^1 one-to-one correspondence on $\Pi(\Gamma_{1,C})$ (this hypothesis is satisfied, for instance, when $\Gamma_{\alpha,C}$ is convex and C^1 for $\alpha \in \{1, 2\}$).

Definition 2.3 The outward unit normal vector on $\Gamma_{1,C}$ is denoted by $\vec{\mathbf{n}}_1$ or $\vec{\mathbf{n}}_{\vec{\mathbf{X}}}$.

The outward unit normal vector on $\Gamma_{2,C}$ is denoted by $\vec{\mathbf{n}}_2$.

The outward unit normal vector for the contact condition, denoted by $\vec{\mathbf{n}}_Y$, is chosen to be that of $\Gamma_{2,C}$:

$$\vec{\mathbf{n}}_Y : \begin{array}{l} \Gamma_{1,C} \rightarrow \mathbb{R}^d \\ \vec{\mathbf{X}} \mapsto n_2(\Pi(\vec{\mathbf{X}})). \end{array} \quad (5)$$

The orthonormal basis, denoted by $(\vec{\mathbf{t}}_1, \vec{\mathbf{t}}_2)$, is the contravariant tangential basis vector defined at the point $\vec{\mathbf{Y}} = \Pi(\vec{\mathbf{X}})$ on the master surface.

Definition 2.4 The initial gap g^0 between $\Gamma_{1,C}$ and $\Gamma_{2,C}$ is defined as the distance function :

$$g^0 : \begin{array}{l} \Gamma_{1,C} \rightarrow \mathbb{R} \\ \vec{\mathbf{X}} \mapsto (\vec{\mathbf{X}}_1 - \Pi(\vec{\mathbf{X}}_1)) \cdot \vec{\mathbf{n}}_Y. \end{array} \quad (6)$$

Remark 2.2 The gap function, corresponding to ray-tracing with respect to a point x , is defined by :

$$g : \begin{array}{l} \Gamma_{1,C} \rightarrow \mathbb{R} \\ \vec{\mathbf{X}} \mapsto (\vec{\mathbf{X}}_1 - \Pi(\vec{\mathbf{X}}_1)) \cdot \vec{\mathbf{n}}_1. \end{array} \quad (7)$$

Definition 2.5 For a displacement field $\vec{\mathbf{u}} = (\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2)$ defined on $\Omega_1 \times \Omega_2$, the normal jump on the slave boundary Γ_1 for the normal displacement is defined as follows :

$$[[\vec{\mathbf{u}} \cdot \vec{\mathbf{n}}]] = (\vec{\mathbf{u}}_2 \circ \Pi - \vec{\mathbf{u}}_1) \cdot \vec{\mathbf{n}}. \quad (8)$$

Concerning the normal stress, we define

$$\boldsymbol{\sigma}(\vec{\mathbf{u}}_1) \vec{\mathbf{n}}_1 = -\boldsymbol{\sigma}_n(\vec{\mathbf{u}}_1) \vec{\mathbf{n}}_1 + \boldsymbol{\sigma}_t(\vec{\mathbf{u}}_1) \quad \text{with } \boldsymbol{\sigma}_n(v_1) = -\boldsymbol{\sigma}(\vec{\mathbf{u}}_1) n_1 \cdot \vec{\mathbf{n}}$$

where $\vec{\mathbf{n}}_1$ is the unit normal vector defined at point $\vec{\mathbf{X}}$ (in the discretized configuration, its orientation depends on the orientation of the node numbering for each element), and $\boldsymbol{\sigma}_t(\vec{\mathbf{u}}_1)$ is the frictional traction applied to the master surface by the point $\vec{\mathbf{X}}$ on the slave surface.

Furthermore,

$$\boldsymbol{\sigma}(\vec{\mathbf{u}}_2 \circ \Pi) n_2 \circ \Pi = \boldsymbol{\sigma}_n(\vec{\mathbf{u}}_2 \circ \Pi) \vec{\mathbf{n}} + \boldsymbol{\sigma}_t(\vec{\mathbf{u}}_2 \circ \Pi) \quad \text{with } \boldsymbol{\sigma}_n(\delta \vec{\mathbf{u}}_2 \circ \Pi) = \boldsymbol{\sigma}(\vec{\mathbf{u}}_2 \circ \Pi) n_2 \circ \Pi \cdot \vec{\mathbf{n}}.$$

Definition 2.6 This allows us to define the normal stress jump as

$$[[\boldsymbol{\sigma}(\bar{\mathbf{u}})\bar{\mathbf{n}}]] = \boldsymbol{\sigma}(\bar{\mathbf{u}}_1)n_1 + \boldsymbol{\sigma}(\bar{\mathbf{u}}_2 \circ \Pi)n_2 \circ \Pi |det(J_\Pi)|, \quad (9)$$

with J_Π denoting the Jacobian matrix of Π .

With these jumps defined, the unilateral frictional contact conditions can be expressed on the slave boundary $\Gamma_{1,C}$ as follows :

$$\left\{ \begin{array}{ll} [[\bar{\mathbf{u}} \cdot \bar{\mathbf{n}}]] \leq g & (i), \\ \boldsymbol{\sigma}_n(\bar{\mathbf{u}}_1) \leq 0 & (ii), \\ \boldsymbol{\sigma}_n(\bar{\mathbf{u}}_1)([[\bar{\mathbf{u}} \cdot \bar{\mathbf{n}}]] - g) = 0 & (iii), \\ [[\boldsymbol{\sigma}(\bar{\mathbf{u}})\bar{\mathbf{n}}]] = 0 & (iv), \\ \|\boldsymbol{\sigma}_t(\bar{\mathbf{u}})\| \leq -F\boldsymbol{\sigma}_n(\bar{\mathbf{u}}) \text{ if } d_t = 0, & (v). \\ \boldsymbol{\sigma}_t(\bar{\mathbf{u}}) = F\boldsymbol{\sigma}_n(\bar{\mathbf{u}}) \frac{d_t}{\|d_t\|} \text{ if } d_t \neq 0 & (vi). \end{array} \right. \quad (10)$$

Equations (10)(v) and (vi) represent the Coulomb friction conditions. To formulate these friction conditions, a coefficient of friction, denoted by $F \geq 0$, is necessary, along with a rigorous notion of sliding velocity. However, in a quasi-static evolution context, instead of a sliding velocity, we use a tangential displacement increment, denoted by d_t . Duvaut and Lions (1972) describe the expression $d_t(u_1, u_2) = (I - n \otimes n)[[u]]$, which, despite being somewhat artificial, exhibits the same characteristics as those obtained for an expression of d_t derived from a time discretization, expressed as :

$$d_t(u_1, u_2) = (I - n \otimes n)([[u]] - [[u^0]])$$

where $[[u^0]]$ is the displacement jump at the previous time step.

Now, let us introduce the Hilbert space V and the convex cone K of admissible displacements :

$$V := H^1(\Omega_1)^d \times H^1(\Omega_2)^d,$$

$$K := \{ \delta \bar{\mathbf{u}} = (\delta \bar{\mathbf{u}}_1, \delta \bar{\mathbf{u}}_2) \in V \mid \delta \bar{\mathbf{u}}_1 = \bar{\mathbf{u}}_{1,D} \text{ on } \Gamma_{1,D} \text{ and } \delta \bar{\mathbf{u}}_2 = \bar{\mathbf{u}}_{2,D} \text{ on } \Gamma_{2,D} \mid [[\delta \bar{\mathbf{u}} \cdot \bar{\mathbf{n}}]] - g \leq 0 \text{ on } \Gamma_{1,C} \}.$$

We assume that $\vec{\mathbf{f}} = (\vec{\mathbf{f}}_1, \vec{\mathbf{f}}_2)$ belongs to $L^2(\Omega_1)^d \times L^2(\Omega_2)^d$, $\vec{\boldsymbol{\ell}} = (\vec{\boldsymbol{\ell}}_1, \vec{\boldsymbol{\ell}}_2)$ belongs to $L^2(\Gamma_{1,N})^d \times L^2(\Gamma_{2,N})^d$, and $\bar{\mathbf{u}}_D$ belongs to $H^{\frac{3}{2}}(\Gamma_{1,D})^d \times H^{\frac{3}{2}}(\Gamma_{2,D})^d$.

Definition 2.7 We introduce a mixed formulation of the elasto-plastic two-body contact problem and project the stresses onto the admissible set by the plastic projector :

$$\mathcal{P}(\boldsymbol{\tau}) := \left\{ \begin{array}{ll} \boldsymbol{\tau} & \text{if } |\boldsymbol{\tau}^{dev}|_F \leq \boldsymbol{\sigma}_\alpha^0 \\ \left(\frac{\xi_{\alpha,iso}}{2G_\alpha + \xi_{\alpha,iso}} \right) + \left(1 - \frac{\xi_{\alpha,iso}}{2G_\alpha + \xi_{\alpha,iso}} \right) \frac{\boldsymbol{\sigma}_\alpha^0}{|\boldsymbol{\tau}^{dev}|_F} \boldsymbol{\tau}^{dev} + \frac{1}{d} Tr(\boldsymbol{\tau}) \mathbb{I}_{d \times d} & \text{if } |\boldsymbol{\tau}^{dev}|_F \geq \boldsymbol{\sigma}_\alpha^0 \end{array} \right. \quad (11)$$

with the shear modulus G_α of the α -th body material.

We consider a test function $\delta \bar{\mathbf{u}}$ in the space of all (smooth) admissible variations of $\bar{\mathbf{u}}$ satisfying possibly homogeneous Dirichlet/Neumann conditions on the appropriate part of $\partial\Omega$.

Definition 2.8 We define the semi-linear form $\mathcal{W}^{int}(\cdot, \cdot)$, called the internal virtual work (associated to the internal energy \mathcal{E}^{int}), the linear form $\mathcal{W}^{ext}(\cdot)$, called the external virtual work, and we denote by $\mathcal{W}^{cb}(\cdot, \cdot)$ the contact (including Dirichlet boundary conditions) virtual work :

1. or a virtual displacement $\bar{\mathbf{u}}^* = \bar{\mathbf{u}} + \delta \bar{\mathbf{u}}$, with operator δ verifying the following properties :

$$\delta \delta \bar{\mathbf{u}} = 0, \quad \delta(\nabla \bar{\mathbf{u}}) = \nabla \delta \bar{\mathbf{u}}, \quad \delta \int_\Omega \bar{\mathbf{u}} d\Omega = \int_\Omega \delta \bar{\mathbf{u}} d\Omega, \quad \delta \bar{\mathbf{u}} = 0 \text{ on } \Gamma_D.$$

$$\mathcal{W}^{int}(\bar{\mathbf{u}}, \delta\bar{\mathbf{u}}) := \sum_{\alpha=1,2} \int_{\Omega_\alpha} \boldsymbol{\sigma}(\bar{\mathbf{u}}_\alpha) : \boldsymbol{\varepsilon}(\delta\bar{\mathbf{u}}_\alpha) d\Omega = \sum_{\alpha=1,2} \int_{\Omega_\alpha} \mathcal{P}(\mathbf{H}_\alpha^{-1} \boldsymbol{\varepsilon}(\bar{\mathbf{u}}_\alpha)) : \boldsymbol{\varepsilon}(\delta\bar{\mathbf{u}}_\alpha) d\Omega = \delta\mathcal{E}^{int}(\bar{\mathbf{u}}) = \partial_{\bar{\mathbf{u}}} \mathcal{E}^{int}(\delta\bar{\mathbf{u}}) \quad (12)$$

$$\mathcal{W}^{ext}(\delta\bar{\mathbf{u}}) := \sum_{\alpha=1,2} \int_{\Omega_\alpha} \vec{\mathbf{f}}_i \delta\bar{\mathbf{u}}_\alpha d\Omega + \sum_{\alpha=1,2} \int_{\Gamma_{\alpha,N}} \vec{\ell}_\alpha \delta\bar{\mathbf{u}}_\alpha d\Gamma \quad (13)$$

$$\mathcal{W}^{cb} := \mathcal{W}^c + \mathcal{W}^b \text{ with } \mathcal{W}^c(\bar{\mathbf{u}}, \delta\bar{\mathbf{u}}) = \int_{\Gamma_{1,C}} \sigma_n(\bar{\mathbf{u}}_1) [[\delta\bar{\mathbf{u}} \cdot \vec{\mathbf{n}}]] d\Gamma.$$

Following [2] and [7], one can derive for the general elasto-plastic case by replacing the contact and frictional constraints, leading to a mixed formulation of the form :

$$\left\{ \begin{array}{ll} \text{Find } (\bar{\mathbf{u}}, \vec{\lambda}_n, \vec{\lambda}_t) \in V \times \Lambda_n \times \Lambda_t \text{ such that} & \\ \mathcal{W}^{int}(\bar{\mathbf{u}}, \delta\bar{\mathbf{u}}) + \mathcal{B}_n(\vec{\lambda}_n, \delta\bar{\mathbf{u}}) + \mathcal{B}_t(\vec{\lambda}_t, \delta\bar{\mathbf{u}}) \geq \mathcal{W}^{ext}(\delta\bar{\mathbf{u}} - \bar{\mathbf{u}}) & \forall \delta\bar{\mathbf{u}} \in V \\ \mathcal{B}_n(\vec{\mu}_n, \vec{\lambda}_n, \bar{\mathbf{u}}) + \mathcal{B}_t(\vec{\mu}_t, \vec{\lambda}_t, \bar{\mathbf{u}}) \geq \langle \vec{\mu}_n, \vec{\lambda}_n, g \rangle_{\Lambda_n} & \forall (\vec{\mu}_n, \vec{\mu}_t) \in \Lambda_n \times \Lambda_t. \end{array} \right. \quad (14)$$

Here, the dual normal cone and the frictional tangential cone are defined as follows : :

$$\Lambda_n := \{\mu \in H^{1/2}(\Gamma_{1C}) / \mu \leq 0 \text{ a.e.}\}' \quad \text{and} \quad \Lambda_t := \{\mu \in H^{1/2}(\Gamma_{1C}) / \|\mu\|_{H^{1/2}} \leq F \text{ a.e.}\}$$

where $\langle \cdot, \cdot \rangle_{\Lambda_n}$ denotes the dual pairing on Λ_n . The bilinear forms that incorporate the contact and frictional conditions are defined by

$$\mathcal{B}_n : \Lambda_n \times V \longrightarrow \mathbb{R}, \quad \mathcal{B}_n(\vec{\mu}, \vec{\mathbf{w}}) := \langle \vec{\mu}, [[\vec{\mathbf{w}}]]_n \rangle_{\Lambda_n}, \quad (15)$$

$$\mathcal{B}_t : \Lambda_t \times V \longrightarrow \mathbb{R}, \quad \mathcal{B}_t(\vec{\mu}, \vec{\mathbf{w}}) := \langle \vec{\mu}, F [[\vec{\mathbf{w}}]]_t \rangle_{L^2(\Gamma_{1C})}. \quad (16)$$

3 Nitsche's Method Formulations

Augmented Lagrangians and Lagrangians are constrained optimization tools that were naturally applied by Rockafellar (1974-1976) to contact problems involving deformable solids. The augmented Lagrangian method has since become widely established for the approximation and resolution of contact problems in both small and large strains, mainly following the research of Curnier and Alart (1988-1991) and Simo and Laursen (1992). The method proposed by Nitsche (1971) has only recently been extended to contact conditions, with or without friction, by Chouly and Hild (2013). The close connection between Nitsche's method and Lagrangian methods is quite clear, and the objective of [8] is to shed some light on this relationship. In this section, assuming that both the solution $\bar{\mathbf{u}}$ and the test functions $\delta\bar{\mathbf{u}}$ are sufficiently regular, we derive from the equilibrium equations and Green's formula :

$$\mathcal{W}^{int}(\bar{\mathbf{u}}, \delta\bar{\mathbf{u}}) - \sum_{\alpha=1,2} \int_{\Gamma_{\alpha,D}} \sigma_n(\bar{\mathbf{u}}_\alpha) \vec{\mathbf{n}}_\alpha \cdot \delta\bar{\mathbf{u}}_\alpha d\Gamma - \int_{\Gamma_{1,C}} \sigma_n(\bar{\mathbf{u}}_1) [[\delta\bar{\mathbf{u}} \cdot \vec{\mathbf{n}}]] d\Gamma = \mathcal{W}^{ext}(\delta\bar{\mathbf{u}}). \quad (17)$$

3.1 Nitsche's Formulation of General Elasto-Plastic Constitutive Law with Friction

Let $\theta \in \mathbb{R}$ be a fixed parameter used to recover different variants of the Nitsche method, as in the linear elastic setting (see article [4]). With the splitting

$$[[\bar{\mathbf{u}} \cdot \vec{\mathbf{n}}]] = ([[\bar{\mathbf{u}} \cdot \vec{\mathbf{n}}]] - \theta \gamma \mathcal{D} \sigma_n(\bar{\mathbf{u}})(\delta u)) + \theta \gamma \mathcal{D} \sigma_n(\bar{\mathbf{u}})(\delta u), \quad (18)$$

As presented in [5] for Tresca friction, we could reformulate the Coulomb friction condition using the projection $\mathbf{P}_{\mathbb{B}(0,\tau)}$. In fact, for a given positive function γ , the friction condition is equivalent to the non-smooth equation :

$$\boldsymbol{\sigma}_{\vec{\mathbf{t}}}(\bar{\mathbf{u}}) = P_{\mathbb{B}(-F \sigma_n(\bar{\mathbf{u}}))}(\boldsymbol{\sigma}_{\vec{\mathbf{t}}}(\bar{\mathbf{u}}) - \gamma \dot{\bar{\mathbf{u}}}). \quad (19)$$

To simplify notations, let use denote by $\mathbf{P}_{n,F}$ the map corresponding to a pair of projections :

$$\mathbf{P}_{n,F}(\vec{x}) = -(\vec{x} \cdot \vec{n})_- \vec{n} + \mathbf{P}_{\mathbb{B}(0,F(\vec{x} \cdot \vec{n})_-)}(\vec{x} - (\vec{x} \cdot \vec{n}) \vec{n}). \quad (20)$$

This application projects the normal part of x onto \mathbb{R}_- and the tangential part onto the ball \mathbb{B} of center 0 and radius $F(\vec{x} \cdot \vec{n})_-$, where F is the friction coefficient.

As a result, contact and friction conditions, in the case of projection, are formulated as :

$$\sigma_n(\vec{u}) = \mathbf{P}_{n,F}(\sigma_n(u) - \frac{Qu}{\gamma} + \frac{g}{\gamma}n + \frac{qw_T}{\gamma}). \quad (21)$$

and Q is the $ndim \times ndim$ matrix $Q := \alpha I_{ndim} + (1 - \alpha)\vec{n}\vec{n}^\top$. We shall consider that the sliding velocity is approximated by $q(u_T - w_T)$, where the expression of q and w_T depend on the time integration scheme used. Using this operator, we obtain the equilibrium equation :

$$\left\{ \begin{aligned} & \mathcal{W}^{int}(\vec{u}, \delta\vec{u}) - \sum_{\alpha=1,2} \int_{\Gamma_{\alpha,D}} \sigma_n(\vec{u}_\alpha) \vec{n}_\alpha \cdot \delta\vec{u}_\alpha \, d\Gamma - \int_{\Gamma_{C1}} \theta \gamma \sigma_n(u_1) \cdot \mathcal{D}\sigma_n(u_1)[v] \, d\Gamma \\ & + \int_{\Gamma_{C1}} \gamma \mathbf{P}_{n,F}(\sigma_n(u_1) - \frac{Qu}{\gamma} + \frac{g}{\gamma}n + \frac{qw_T}{\gamma}) \cdot (\theta \mathcal{D}\sigma_n(u_1)[v] - \frac{v}{\gamma}) \, d\Gamma = \mathcal{W}^{ext}(\delta\vec{u}). \end{aligned} \right. \quad (22)$$

Remark 3.1 For $\theta \neq 0$, the tangent system requires complex second-order derivatives, making the non-symmetric variant ($\theta = 0$) appealing for simplicity. The symmetric variant ($\theta = 1$) derives from a potential and yields symmetric tangent systems, while the skew-symmetric variant ($\theta = -1$) offers robustness regarding the Nitsche parameter γ [3, 1, 4, 2].

Using this, Nitsche's contact term, corresponding to the virtual contact work for contact with friction, reads as :

$$\mathcal{W}^c = - \int_{\Gamma_{C1}} \theta \gamma \sigma_n(u_1) \cdot \mathcal{D}\sigma_n(u_1)[v] \, d\Gamma + \int_{\Gamma_{C1}} \gamma \mathbf{P}_{n,F}(\sigma_n(u_1) - \frac{Qu}{\gamma} + \frac{g}{\gamma}n + \frac{qw_T}{\gamma}) \cdot (\theta \mathcal{D}\sigma_n(u_1)[v] - \frac{v}{\gamma}) \, d\Gamma.$$

where Γ_C is the contact boundary, $\delta\vec{u}$ a test fonction.

Remark 3.2 In [7], the Nitsche method for frictional contact guarantees existence/uniqueness in dynamics for any parameters. In statics, existence requires a large Nitsche parameter, while uniqueness depends on a mesh-sensitive friction bound.

Remark 3.3 Unlike penalty methods which allow penetration and suffer from ill-conditioning, Nitsche's method better enforces non-penetration and reduces spurious stress oscillations in friction problems [8].

Remark 3.4 The parameter γ plays a similar role across penalty, augmented Lagrangian, and Nitsche methods. Optimal choice follows $\gamma = \gamma_0/h$, where γ_0 has elastic modulus dimension (e.g., bulk modulus). For large strains, γ_0 must reflect maximum tangent moduli to ensure coercivity [4, 6, 15, 13].

Remark 3.5 This work employs segment-to-segment integration with contact constraints enforced at each slave element. Following [13], contact integrals are evaluated using Legendre-Gauss quadrature on slave surface faces. Contact detection operates independently at each quadrature point, where pairs are formed between slave points and their projections on master surfaces. For efficiency, candidate master faces are identified using R-tree organized influence boxes with logarithmic complexity [12].

4 Numerical Results

We conclude this note with numerical results aimed at testing different models, utilizing classical finite element discretization of the previously discussed Nitsche method, implemented in SYS-TUS/SYSWELD 2024. Due to lack of space, we only describe here the dudging of a tube. Mechanical tube expansion consists in plastically deforming a tube using four rotating rollers inserted inside the tube, which gradually expand during rotation. After closing the clearance of few mm, the tube comes into

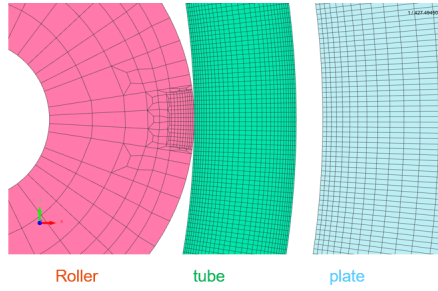


FIGURE 1 – roller, tube and plate

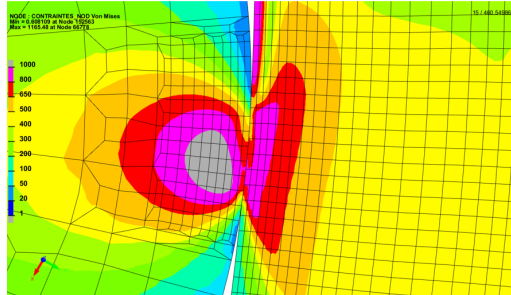


FIGURE 2 – tube Von Mises iso-stress

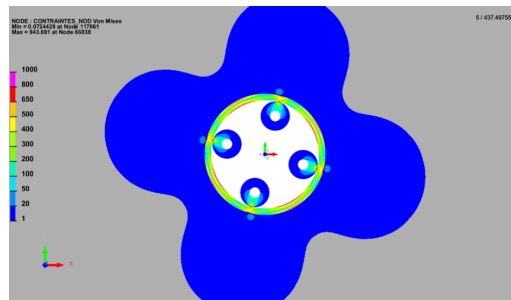


FIGURE 3 – tube and plate : Von Mises Stress

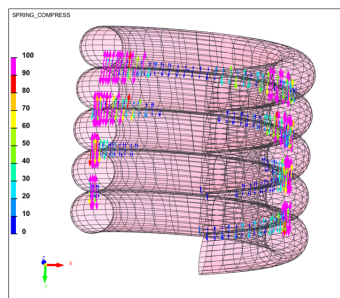


FIGURE 4 – spring compression

contact with the plate and plastically deforms it. At the end of the process, despite the elastic springback, a residual contact pressure between tube and plate remains.

For this type of study, using SYSTUS software with the Nitsche contact is advantageous because we can use of a quadratic mesh for better representation of the geometry and a more accurate stress calculation, with non-coincident meshing of the contact surfaces. The good convergence of calculations with this type of contact is impossible to process with augmented Lagrangian in reasonable computation times, despite our efforts it was impossible to make the resolution methods converge with pure Lagrangian or penalty.

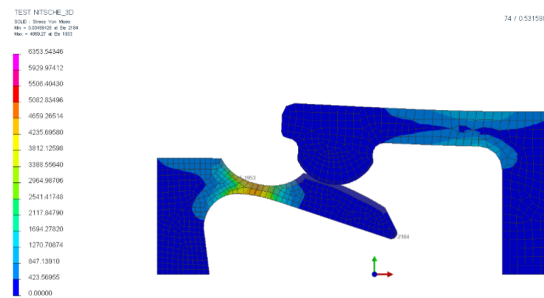


FIGURE 5 – "clip"

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