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A stochastic ALE-approach for rolling contact analysis

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Abstract — The well established arbitrary Lagrangian-Eulerian (ALE) description for the numerical treatment of rolling contact with the finite Element Method (FEM) is extended for considering random input parameters. A multiplicative decomposition approach for high-dimensional stochastic spaces is suggested. The accuracy and efficiency of the proposed method is compared to brute force Monte-Carlo Simulation (MCS) on a simple model problem.
Mots clés — Arbitrary Eulerian Lagrangian, rolling contact, random input parameter, multiplicative decomposition of high-dimensional spaces

1 Introduction

For the efficient computational treatment of rolling contact problems an Arbitrary Lagrangian Eulerian (ALE) description has been established [4]. Besides the basic implementation, special measures must have been considered for the physically consistent treatment of the tractive contact conditions [8] and the treatment of history or temperature dependent constitutive equations, see e.g. [5, 3]. An extension for the simulation of the high-frequency response of rolling tires excited by the road-roughness has been presented in [2] for the investigation of tire noise radiation.

In this contribution the extension of existing methods for the efficient treatment of random loading conditions and material behavior is introduced. Randomization incorporates an additional computational complexity to the problem, as with each random variable a new (infinite) dimension is added. In particular in the case of random-fields the dimensionality may increase drastically, e.g. up to 100 or even more. That so-called curse of dimensions has been investigated in the stochastic finite element society for about two decades now, in order to develop suitable model-order reduction techniques for the numerical treatment of that kind of computational complexity, [1].

Here, we suggest a multiplicative decomposion of variables approach, which has been proven as an effective method to overcome in particular the burden of dimensionality, [7]. After a short introduction into the ALE description of rolling, the randomization of the equations of motion is briefly described, followed by the outline of the treatment of random loadings and material properties. To demonstrate the efficiency of the proposed method, a simple model problem is studied and compared to brute force Monte-Carlo method with regard to accuracy and numerical efficiency.

2 The ALE-approch for rolling contact

The arbitrary Lagrangian Eulerian decomposition of motion has been well established as a powerful tool for rolling contact analysis [4]. Basic idea is the introduction of a reference configuration which describes the pure rigid body motion in an Eulerian picture, as depicted in fig. 1.

By that, the deformation gradient is written as multiplicative decomposition

$$\boldsymbol{F} = \boldsymbol{F}_{\phi} \cdot \boldsymbol{F}_{\chi} \,, \tag{1}$$

and the material time derivative is decomposed into a relative and convective part,

$$\frac{\mathrm{d}\hat{\phi}}{\mathrm{d}t} = \frac{\partial\hat{\phi}}{\partial t} + \mathrm{Grad}\hat{\phi} \cdot \dot{\chi} \,. \tag{2}$$

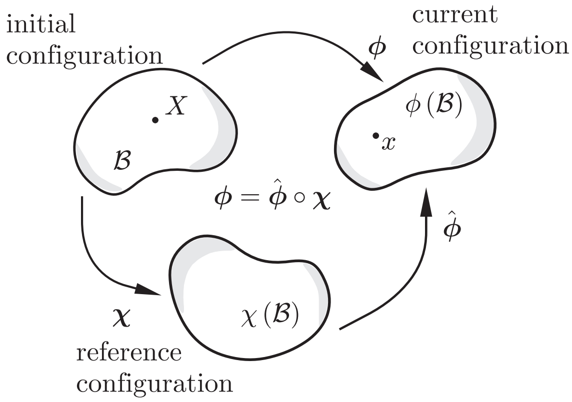

Figure 1: The ALE-decomposition of motion.

That composition is characterized by:

- mesh-points are neither fixed to space nor to the material points
- surfaces move like the material boundary

In the context of finite element analysis this leads to the following advantages:

- stationary rolling is described time-independent, i.e. can be treated in a quasi-static manner
- the fine spatial discretization may be concentrated to the contact area with large deformation gradients, by that much less unknowns are necessary.

In contrast, additional effort is needed to trace the material history of the particles during the path within the spatially fixed mesh and the treatment of the tangential contact conditions, see [3, 5, 8].

For the efficient implementation into the finite element method (FEM) some reformulations of the weak form of the equation of motion are needed in order to eliminate second order gradients, i.e. to avoid a C^1-smooth finite element approximation, for details it is referred to [4]. Finally one ends up with a non-linear quasi-static formulation as

$$[\boldsymbol{K} - \boldsymbol{K}_c - \boldsymbol{W}]\Delta\boldsymbol{u} = \boldsymbol{f}_{\text{ext}} + \boldsymbol{f}_{\text{inertia}} + \boldsymbol{f}_{\text{cont}} - \boldsymbol{f}_{\text{int}} \tag{3}$$

where $\boldsymbol{K}$ is the structural stiffness matrix, $\boldsymbol{K}_c$ results from the linearization of the contact forces, $\boldsymbol{W}$ is the ALE inertia matrix and $\Delta\boldsymbol{u}$ is the increment of the nodal displacements. Further, $\boldsymbol{f}_{\text{ext}}$ are the externally applied forces, $\boldsymbol{f}_{\text{inertia}}$ are the ALE inertia forces, $\boldsymbol{f}_{\text{cont}}$ are the contact forces and $\boldsymbol{f}_{\text{int}}$ are the internal forces.

3 Randomization

Considering stochastic input, the finite element system (3) becomes dependent from the input random parameters θ as well as the spatial coordinates $\boldsymbol{x}$, thus, it is changed to a Stochastic Finite Element Method (SFEM).

In this presentation we will go for the description of stochastic material properties and loadings. Loading conditions are described as uniform distributed random variables, while the material properties are modeled as Gaussian random fields via Karhunen-Loeve Expansion (KLE)

$$E(\boldsymbol{x},\theta) = E_0 + \sum_{i=1}^{r} \xi_i(\theta)\sqrt{\kappa_i}E_i(\boldsymbol{x}). \tag{4}$$

Here, $\xi_i(\theta)$ are random variables, $\sqrt{\kappa_i}$ are the eigenvalues of the underlying solution of the Fredholm integral equation and $E_i(x)$ are the corresponding eigen-vectors. An exponential covariance function has been assumed and the correlation length has been chosen artificially in order to investigate the solution method in dependency of the stochastic dimension.

4 Solution Scheme

For the solution of the linearized random finite element equation (3) an incremental scheme is applied,

$$\boldsymbol{u}_k(\theta) = \boldsymbol{u}_{k-1}(\theta) + \Delta\boldsymbol{u}_k(\theta) \in \mathbb{R}^n, \tag{5}$$

where $\Delta\boldsymbol{u}_k(\theta)$ is the incremental displacement update in step k. The displacement increment is approximated by

$$\begin{aligned}\Delta\boldsymbol{u}_k(\theta) &= [\lambda_{k,1}(\theta)\mathbf{Z}_1 + \lambda_{k,2}(\theta)\mathbf{Z}_2]\,\mathbf{d}_k, &(6a)\\ &= [\mathbf{Z}_1\mathbf{d}_k, \mathbf{Z}_2\mathbf{d}_k]\,\boldsymbol{\lambda}_k(\theta), &(6b)\end{aligned}$$

where $\mathbf{d}_k$ are deterministic vectors and $\lambda_{k,1}(\theta)$ and $\lambda_{k,2}(\theta)$ are random variables associated to the bulks DOF's and the DOF's assigned to the (potential) contact nodes. The diagonal matrices $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are introduced to separate the non-contact and contact nodes.

Thus, we are seeking for the triplets $[\lambda_{k,1}(\theta), \lambda_{k,2}(\theta), \mathbf{d}_k]$ which we solve by a staggered greedy iteration. First we solve for the unknown deterministic vector

$$\widetilde{\mathbf{K}}_k\mathbf{d}_k = \widetilde{\mathbf{F}}_k, \tag{7}$$

with the matrices

$$\begin{aligned}\widetilde{\mathbf{K}}_k &= \mathbb{E}\left\{[\lambda_{k,1}(\theta)\mathbf{Z}_1 + \lambda_{k,2}(\theta)\mathbf{Z}_2]^{\mathrm{T}}\left[\mathbf{K}(\theta) + \alpha\mathbf{K}_{\mathrm{p}}(\theta)\right][\lambda_{k,1}(\theta)\mathbf{Z}_1 + \lambda_{k,2}(\theta)\mathbf{Z}_2]\right\} &(8a)\\ \widetilde{\mathbf{F}}_k &= \mathbb{E}\left\{[\lambda_{k,1}(\theta)\mathbf{Z}_1 + \lambda_{k,2}(\theta)\mathbf{Z}_2]^{\mathrm{T}}\left[\mathbf{F}_k(\theta) + \alpha\mathbf{F}_{\mathrm{p},k}(\theta)\right]\right\} &(8b)\end{aligned}$$

where $\mathbb{E}$ denotes the expectation operator. Once we solved for $\mathbf{d}_k$, which has been orthogonalized via Gram-Schmidt procedure, we solve for the stochastic increments,

$$\left([\mathbf{Z}_1\mathbf{d}_k, \mathbf{Z}_2\mathbf{d}_k]^{\mathrm{T}}\left[\mathbf{K}(\theta) + \alpha\mathbf{K}_{\mathrm{p}}(\theta)\right][\mathbf{Z}_1\mathbf{d}_k, \mathbf{Z}_2\mathbf{d}_k]\right)\boldsymbol{\lambda}_k(\theta) = [\mathbf{Z}_1\mathbf{d}_k, \mathbf{Z}_2\mathbf{d}_k]^{\mathrm{T}}\left[\mathbf{F}_k(\theta) + \alpha\mathbf{F}_{\mathrm{p},k}\right]. \tag{9}$$

The iteration is repeated until the convergence criteria in the displacement increment is reached. For algorithmic details it is referred to [6].

5 Computational Example

To demonstrate the efficiency of the proposed approach a simple 2-D wheel is discretised 1800 linear triangle elements, see fig 2. At the inner ring a stochastic axial loading is applied, which is assumed to be uniform distributed. Further, the angular speed of the rolling wheel is assumed to be uniform distributed. The Young's modulus is described by a two-dimensional Gaussian random field discretized via KLE to end up with high-dimensional stochastic domain. Thus, in total 23 random variables are involved. The contact constraints are enforced via penalty-method.

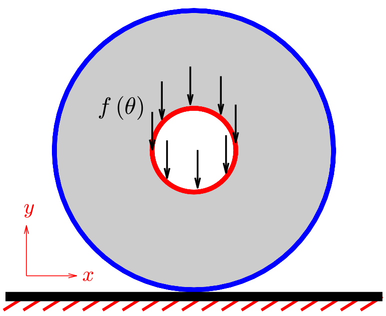

Figure 2: Rolling Wheel.

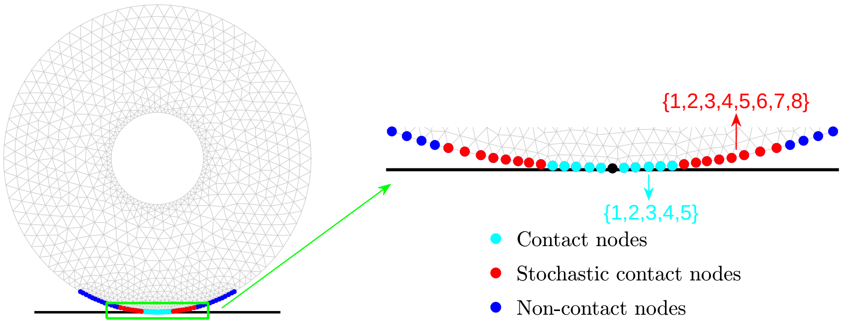

Figure 3: Finite element mesh and contact surface.

The contact nodes are separated into contact nodes and non-contact nodes, which are almost true in contact or not in contact respectively, and stochastic contact-nodes, which by chance may be in contact or not (numbered in fig. 3).

The solution is compared to a native Monte-Carlo-Simulation with 10^4 samples. In our SFEM approach the same number of samples have been used to avoid influence from sampling. Here we will have a detailed view to the small penetration of the contact nodes, which appear due to the penalty approach. In figure 4 the PDF of the stochastic contact node 1 is depicted. A clear discontinuity at the gap-displacement is seen. Further, there is a good agreement between MSC and the SFEM approach suggested here.

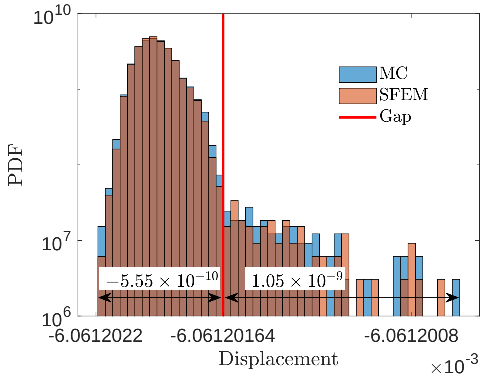

Figure 4: PDF of the stochastic penetration of the stochastic contact node 1.

The convergence of the proposed decomposition approach is depicted in fig. 5. It is seen, that only a few iterations (deterministic vectors and random variables) are needed, to obtain a well converged approximation of the solution. In comparison to MCS a speed-up in computation time of about 27 has been measured for the SFEM approach.

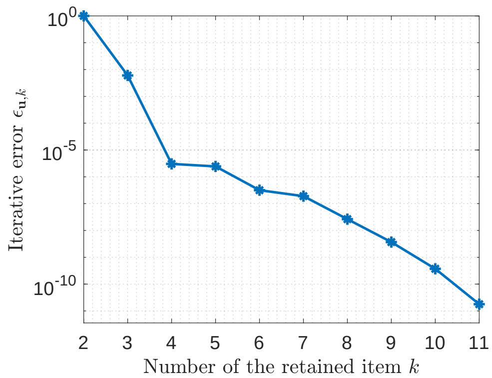

Figure 5: Convergence of the iterative error in dependency of the iteration progress.

6 Summary and Conclusions

In this presentation we extended the well established ALE method for rolling contact for stochastic computations. Loadings and material properties have been considered as random variables or random fields. An efficient multiplicative decomposition scheme has been applied by which the solution is decoupled into the computation of deterministic vectors and random variables, which is solved iteratively by a greedy scheme. As the random variables decompose into scalar equations, quite high-dimensional random spaces can be tackled with less numerical effort and high accuracy.

The solution scheme has been proven to provide very accurate approximations in comparison to MSC with much less numerical effort.

So far, only very simple examples (linear elastic material, small deformation etc.) have been investigated. The consideration of large displacement with non-linear and perhaps inelastic material response will be part of our future work.

References

[1] Chinesta, F., Huerta, A., Rozza, G. and Willcox, K. (2018): Model reduction methods. In: Stein, E., de Borst, R. and Hughes, T. J. R.: Ecyclopedia of Computational Mechanics, Second Edition, Part 1 Solids and Structures, Chapter 3

[2] Brinkmeier M., Nackenhorst U. An approach for large-scale gyroscopic eigenvalue problems with application to high-frequency response of rolling tires (2008) Computational Mechanics, 41 (4), pp. 503 - 515, DOI: 10.1007/s00466-007-0206-6

[3] Garcia M.A., Kaliske M. A consistent viscoelastic formulation for the numerical analysis of steady state rolling tires (2018) International Journal of Plasticity, 101, pp. 24 - 41, DOI: 10.1016/j.ijplas.2017.09.014

[4] Nackenhorst U. The ALE-formulation of bodies in rolling contact Theoretical foundations and finite element approach (2004) Computer Methods in Applied Mechanics and Engineering, 193 (39-41 SPEC. ISS.), pp. 4299 - 4322,DOI: 10.1016/j.cma.2004.01.033

[5] Suwannachit A., Nackenhorst U. A novel approach for thermomechanical analysis of stationary rolling tires within an ALE-kinematic framework (2013) Tire Science and Technology, 41 (3), pp. 174 - 195.

[6] Zheng Z., Nackenhorst U. Stochastic arbitrary Lagrangian–Eulerian formalism for stochastic rolling contact analysis (2025) Computational Mechanics, 76 (5), pp. 1285 - 1302, DOI: 10.1007/s00466-025-02649-7

[7] Zheng Z., Nackenhorst U. Semi-reduced order stochastic finite element methods for solving contact problems with uncertainties (2023) Computational Mechanics, 72 (5), pp. 991 - 1008, DOI: 10.1007/s00466-023-02323-w

[8] Ziefle M., Nackenhorst U. Numerical techniques for rolling rubber wheels: Treatment of inelastic material properties and frictional contact (2008) Computational Mechanics, 42 (3), pp. 337 - 356, DOI: 10.1007/s00466-008-0243-9