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A Continuum Nonlinear Model for Wave Propagation and Quasistatic Flow in Granular Media

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Abstract — Granular materials exhibit complex behaviour, transitioning between solid-, liquid- and gas-like states depending on factors such as compaction, stress and deformation rate. This study proposes a continuous nonlinear model for wave propagation in granular media, inspired by a modified form of the Euler equations and informed by Hertzian contact theory. The model captures the dependence of wave speed on compaction, and deformation tensor invariants are used to extend it and describe shear-induced dilation. Finite-volume simulations demonstrate that the model stabilises granular piles in the quasi-static regime and recovers realistic angles of repose. These results represent a promising step towards the large-scale modeling of granular assemblies, such as railway ballast.

Mots clés — Granular media, nonlinear waves, compaction, continuum modeling, finite volume method.

1 Introduction

Granular media are ubiquitous in both natural and industrial contexts, ranging from geological formations and landslides to pharmaceutical powders, food processing, and railway infrastructure. Despite their everyday presence, they remain notoriously difficult to model, as their macroscopic behavior originates from complex microscopic interactions. One of the most remarkable features of granular materials is their ability to behave as a gas, a fluid, or a solid depending on state variables such as the packing (or filling) fraction, the velocity of the individual particles, the external forcing, and the magnitude and direction of applied stresses [1, 2].

At very low packing fractions and high particle agitation, granular materials behave similarly to a gas, where collisions are binary, momentum-driven, and strongly dissipative. At intermediate packing fractions and deformation rates, they display fluid-like behavior, exhibiting flow phenomena such as shear bands, segregation, and convection. At high packing fractions and low deformation rates, granular media behave as amorphous solids, capable of supporting shear stresses and forming stable structures such as piles and arches. A unifying continuous model should, in principle, be able to describe transitions between these regimes, capturing both fluidization and jamming.

A fundamental source of nonlinearity in granular materials lies in the local force–displacement relationship between individual particles. In many classical models, particles are assumed to be elastic spheres, and their interaction is governed by Hertzian contact theory, which states that the normal force scales as a nonlinear power of the overlap distance between particles. This nonlinearity at the contact scale is responsible for a wide range of emergent phenomena at the macroscopic scale, including amplitude-dependent wave speeds, soliton-like wave propagation, and nonlinear attenuation.

The objective of this work is to develop a continuous nonlinear model capable of describing wave propagation in granular media, with special focus on railway ballast. Ballast materials are composed of coarse aggregates with irregular shapes, typically with diameters ranging from a few to several centimeters. They play a critical role in supporting railway tracks, dissipating loads, and providing drainage. The characteristic length scale of railway lines can extend over hundreds of meters to kilometers, rendering fully discrete simulations extremely expensive or unfeasible.

While the Discrete Element Method (DEM) has emerged as a powerful tool for studying granular physics in controlled volumes, it quickly becomes computationally intractable when scaling up to realistic engineering dimensions. By contrast, a continuous formulation permits the use of grid-based numerical methods, which are more straightforward to parallelise efficiently, thus enabling simulations at a much larger scale.

Several continuous models for granular flow already exist in the literature. Many of them are inspired by classical Navier–Stokes theory but incorporate an effective shear viscosity depending on the so-called inertial number

$$I = \frac{\dot{\gamma} d}{\sqrt{p/\rho}},$$

where $\dot{\gamma}$ is the shear rate, d a characteristic particle diameter, p the pressure, and ρ the bulk density. This dimensionless number provides a measure of the relative importance of inertial forces over confining stresses. Experimental studies in plane shear, chute flow, and annular Couette configurations have been used to identify empirical friction laws as functions of I [1, 2]. These relations allow the definition of an effective viscosity that changes dynamically with the flow regime.

However, a major limitation of such models arises in the quasi-static regime, where $\dot{\gamma} \to 0$ and thus $I \to 0$. In this limit, the effective viscosity vanishes, and the material is no longer able to sustain shear stresses, which contradicts the very nature of a granular solid. As a result, classical continuum models fail to describe equilibrium situations such as stable piles, jammed configurations, and the onset of yielding. Overcoming this limitation is a fundamental motivation of this work, since wave propagation in time must occur on a mechanically stable background. Therefore, the ability to sustain stress in the quasi-static regime is an essential feature that any realistic model of granular media must satisfy.

2 Model formulation and main ideas

2.1 Compaction-modified Euler equations

The starting point of the present model is inspired by the work of Favrie and Gavrilyuk (2013) [3], who proposed a modification of the compressible Euler equations in order to incorporate compaction effects in dense granular or multiphase media. In the classical Euler formulation for a compressible fluid, the system reads

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0,\\
\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p\mathbf{I}) &= 0,\\
\frac{\partial E}{\partial t} + \nabla \cdot ((E + p)\mathbf{u}) &= 0,
\end{aligned}$$

where ρ is the density, $\mathbf{u}$ is the velocity field, p is the pressure, E is the total energy, and $\mathbf{I}$ is the identity tensor.

In the case of granular materials, the key modification lies in the definition of the internal energy. The classical equation of state for a gas is replaced by an energy function that strongly penalizes configurations above a critical volume fraction α_c, corresponding to a maximum packing. When the local volume fraction satisfies $\alpha > \alpha_c$, the energy increases sharply, representing the onset of inter-particle contact and deformation. This approach enables the model to transition smoothly from a dilute regime, where grains are essentially not in contact, to a compact regime, where elastic deformation of particles occurs. The corresponding pressure is obtained through a thermodynamic relationship of the form

$$p = \rho^2 \frac{\partial e}{\partial \rho},$$

where e is the specific internal energy.

2.2 Connection with Hertzian contact theory

The definition of the compaction energy is directly inspired by Hertzian contact theory. In a one-dimensional chain of spherical particles, the contact force F between two adjacent particles is expressed as

$$F = k\, \delta(\alpha)^{3/2},$$

where δ is the overlap distance and k is a constant depending on the Young's modulus and the geometry of the particles. The corresponding potential energy scales as $\delta^{5/2}$. By relating the overlap to the macroscopic volumetric strain, an equivalent form of energy density can be constructed.

$$e(\alpha) = \begin{cases} k\,\delta(\alpha)^{5/2}, & \text{if } \alpha \geq \alpha_c \\ 0 & \text{if } \alpha < \alpha_c \end{cases}$$

This choice allows the present continuum model to recover the nonlinear relationship between wave speed and compaction obtained by Coste (1993) [4] for one-dimensional granular chains. In particular, the characteristic wave speed c is found to depend on the level of compression. This property is recovered naturally by the model, confirming its physical consistency at the macroscopic scale.

A one-dimensional Riemann problem is considered, with an initial discontinuity imposed at $x = 0$, as shown in Figure 1a. The left-hand side corresponds to a compacted granular state, whereas the right-hand side is initially uncompressed. Density, velocity, and contact pressure are reported. Owing to the higher effective stiffness (and hence wave speed) in the compacted phase, a compressive shock wave propagates rightwards into the dilute region, while a rarefaction wave propagates leftwards into the compacted medium. The system is solved using a finite volume method with a MUSCL-H reconstruction and a Rusanov numerical flux [5].

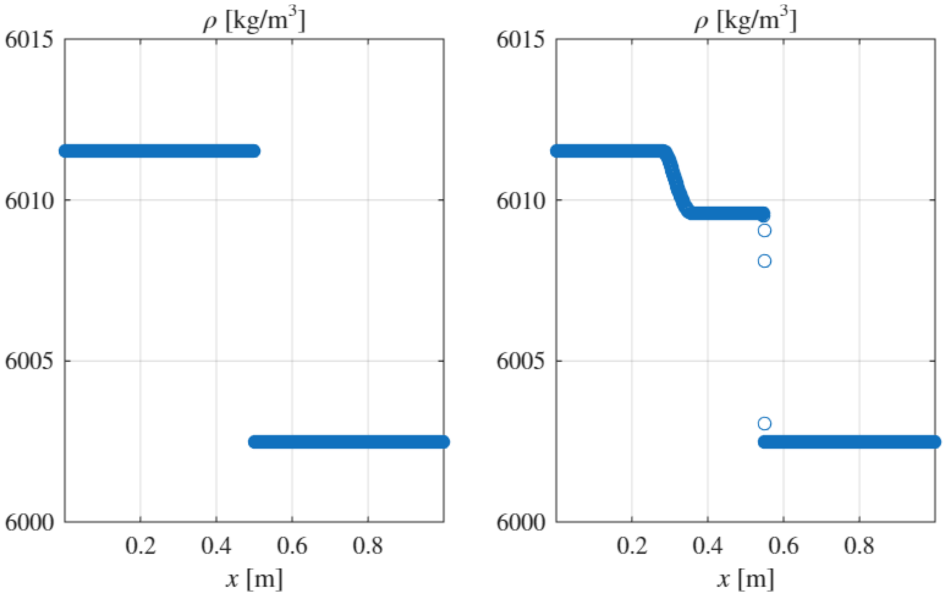

Figure 1: Left, initial state of the Riemann problem. Right, shock and rarefaction waves after 10 ms.

2.3 Extension to two and three dimensions

While the one-dimensional formulation provides a valuable validation benchmark, real granular systems ultimately require a fully two- or three-dimensional description. In higher dimensions, compaction is no longer purely volumetric and is strongly influenced by shear-induced rearrangements. One of the best-known manifestations of this phenomenon is Reynolds dilatancy: when a dense granular assembly is sheared, its volume tends to increase as particles must move around each other.
To capture this geometric effect within a continuum description, we introduce an additional energy term that depends on the invariants of the Finger tensor

$$\mathbf{G} = \mathbf{F}^{-T}\mathbf{F}^{-1},$$

where $\mathbf{F}$ is the deformation gradient tensor. The invariants of $\mathbf{G}$ provide information not only on volumetric changes but also on distortional (shear) deformations.

An energy-free manifold is defined in the space of invariants, corresponding to configurations where particles are not in contact. Once a certain threshold in volume fraction is exceeded — a threshold that itself depends on the amount of shear — the energy sharply increases, mimicking the onset of contact and frictional locking between grains. This threshold is therefore no longer constant, but varies dynamically with the deformation state, in accordance with the expected behaviour of dense granular assemblies.

This formulation is designed to reproduce a key geometric property of granular materials: the necessity of dilation under shear in initially dense configurations. Preliminary numerical experiments suggest that this mechanism can promote the stabilisation of granular piles. However, the resulting system is not strictly hyperbolic in all regions of the deformation space, which raises important questions concerning its mathematical well-posedness and its numerical treatment. These aspects, together with the current limitations of the model and the strategies under consideration to address them, will be discussed in detail during the presentation.

3 Conclusions

This work presents the foundations of a continuous nonlinear model aimed at describing both dynamic and quasi-static behaviour in granular media. By introducing a compaction-based modification of the Euler equations and defining a contact energy inspired by Hertzian mechanics, the model is able to reproduce key features of nonlinear wave propagation in one-dimensional granular chains. The proposed extension to higher dimensions, based on the use of deformation tensor invariants, is designed to capture essential geometric effects such as shear-induced dilation.

Preliminary results in the quasi-static regime indicate that the model can promote the stabilisation of a granular pile under gravity, without the introduction of an explicit viscosity term. This represents an encouraging step towards overcoming the well-known limitations of classical continuum models in the low strain-rate limit. Nevertheless, a central theoretical and numerical challenge remains: the governing system is not strictly hyperbolic in its current formulation. As a consequence, the problem may become ill-posed in certain regions of the deformation space, and the reliability of standard shock-capturing schemes is not guaranteed. Further modifications of the energy function, additional constraints, or the introduction of suitable regularisation strategies may therefore be necessary. Alternatively, different classes of numerical methods, possibly inspired by solid mechanics or mixed formulations, may be required to fully exploit the potential of the proposed approach.

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